Development of $\underline{sn}\, x$, $\underline{cn}\, x$, $\underline{dn}\, x$, by means of their addition theorems.

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Taking the three addition theorems and clearing away the fractions,

$$sn\ \overline{x+y} = \begin{cases} \sin x\, cn\, y\, dn\, y + sn\, y\, cn\, x\, dn\, x \\ + k^2 \sin^2 x \sin^2 y \sin \overline{x+y} \end{cases} \tag{1}$$

$$cn\ \overline{x+y} = \begin{cases} cn\, x\, cn\, y - sn\, x \sin y\, dn\, x\, dn\, y \\ + k^2 \sin^2 x \sin^2 y \cos \overline{x+y} \end{cases} \tag{2}$$

$$dn\ \overline{x+y} = \begin{cases} dn\, x\, dn\, y - k^2\, sn\, x \sin y\, cn\, x\, cn\, y \\ + k^2 \sin^2 x \sin^2 y\, dn\, \overline{x+y} \end{cases} \tag{3}$$

Let

$$\begin{aligned} sn\, x &= a_1 x + a_3 x^3 + a_5 x^5 + \dots \\ cn\, x &= a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \dots \\ dn\, x &= b_0 + b_2 x^2 + b_4 x^4 + b_6 x^6 + \dots \end{aligned}$$

Substitute in (1) (2) (3) the expansions for $\sin x$, $\sin y$, $\sin \overline{x+y}$ $\cos x$, $\cos y$, $\cos \overline{x+y}$, $dn\, x$, $dn\, y$, $dn\, \overline{x+y}$, and then pick out the coefficients of y in (1), (2), (3). Then equate the like powers of x in the resulting series.

From (1)

$$\begin{aligned} a_1 &= a_0 b_0 a_1 \\ 3a_3 &= a_1(a_0 b_2 + a_2 b_0) \\ 5a_5 &= a(a_0 b_4 + a_2 b_2 + a_4 b_0) \end{aligned}$$

and so on.

From (2)

$$\begin{aligned} a_0 &= a_0^2 \\ -2a_2 &= a_1b_0(a_1b_0) \\ -4a_4 &= a_1b_0(a_1b_2 + a_3b^0) \\ -6a_6 &= a_1b_0(a_1b_4 + a_3b_2 + a_5b_0) \end{aligned}$$

and so on.

From (3)

$$\begin{aligned} b_0 &= b_0^2 \\ -2b_2 &= k^2a_1a_0(a_1a_0) \\ -4b_4 &= k^2a_1a_0(a_1a_2 + a_3a_0) \\ -6b_6 &= k^2a_1a_0(a_1a_4 + a_3a_2 + a_5a_0) \end{aligned}$$

and so on.

Hence $a_0 = 1,\quad b_0 = 1,\quad a_1$ *undetermined,*

$$a_2 = -\frac{a_1^2}{1.2},\quad b_2 = -\frac{k^2a_1^2}{1.2},$$

$$a_3 = \tfrac{1}{3}a_1(a_2 + 6_2) = \frac{-a_1^3(1+k^2)}{1.2.3}$$

Similarly $$a_4 = \frac{a_1^4(1+4k^2)}{1.2.3.4},\quad b_4 = \frac{a_1^4k^2(4+k^2)}{1.2.3.4},$$

$$a_5 = \tfrac{1}{5}a_1(b_4 + a_2b_2 + a_4)a_1^5 = \frac{1+14k^2+k^4}{1.2.3.4.5},$$

and so on.